Webs, foams, knot invariants, and representation theory

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Illustrating Number Theory and Algebra
ICERM
October 21, 2019
Overview

1. Knots and their (polynomial) invariants

2. Webs and representation theory

3. Knot homologies and foams

4. Some illustrative consequences
Knots and topology

- A knot is precisely what you think it is: a flexible, closed, knotted piece of string in three-dimensional space.

- Although the study of knots at first appears to be a rather niche problem in topology (i.e. the study of how 1d spaces embed in a certain 3d space), knots provide a means to study 3d topology.

- E.g. the Lickorish-Wallace Theorem states that any closed, orientable, connected 3-manifold can be obtained via surgery on a knot/link.
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Knots and topology

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Diagrams for knots

- Despite knots (and links) being inherently 3-dimensional objects, they can be studied via their 2-dimensional diagrams:

Theorem (Reidemeister, 1927)

There is a bijection from the set of knots to the set of equivalence classes of knot diagrams under the Reidemeister moves RI, RII, and RIII.
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![Knot diagrams example](attachment:image_url)

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RI: \[ \sim \quad \sim \],

RII: \[ \sim \quad \sim \]

RIII: \[ \sim \quad \sim \]
Knot invariants and the Jones polynomial

- Until the 1980’s, (most) invariants of knots did not make use of the diagrammatic description afforded by the Reidemeister Theorem, instead being given in terms of “classical” constructions in algebraic topology (e.g. fundamental group, covering spaces, homology).
- In 1985, Jones introduced a polynomial invariant $V_q(K) \in \mathbb{Z}[q, q^{-1}]$ for knots $K \subset S^3$ using algebraic methods (braid group representations).
- Kauffman reformulated the Jones polynomial in diagrammatic terms:

$$
\begin{align*}
\begin{array}{c}
\curvearrowright \\
\curvearrowright \\
\curvearrowleft \\
\end{array} & = q - q^{-1} \\
\begin{array}{c}
\curvearrowright \\
\curvearrowright \\
\\end{array} & = -q \\
\begin{array}{c}
\curvearrowright \\
\\end{array} & = \bigcirc = [2] := q + q^{-1}
\end{align*}
$$

i.e. as a function from the set of (oriented) knot diagrams to $\mathbb{Z}[q, q^{-1}]$ that is invariant under the Reidemeister moves.
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\downarrow \\
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\left\langle \begin{array}{c}
\uparrow \\
\downarrow \\
\uparrow \\
\end{array} \right\rangle &= -q \\
\bigcirc &= [2] := q + q^{-1}
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\[
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\begin{pmatrix} \downarrow \vphantom{2} \\ \uparrow \vphantom{2} \end{pmatrix} &= \chi - q^{-1} \\
\begin{pmatrix} \downarrow \vphantom{2} \\ \uparrow \vphantom{2} \end{pmatrix} &= -q \\
\begin{pmatrix} \bigcirc \vphantom{2} \end{pmatrix} &= [2] := q + q^{-1}
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\begin{align*}
\begin{array}{c}
\xymatrix{\ar@/^/[r] & \ar@/_/[r]}
\end{array}
\end{align*}
\quad = \quad 
\begin{array}{c}
\xymatrix{\ar@/^/[r] & \ar@/^/[r]}
\end{array}
- q^{-1}
, \\
\begin{array}{c}
\xymatrix{\ar@/^/[r]}
\end{array}
\begin{array}{c}
\xymatrix{\ar@/^/[r]}
\end{array}
\quad = \quad -q
, \\
\begin{array}{c}
\xymatrix{\ar@/^/[r]}
\end{array}
\begin{array}{c}
\xymatrix{\ar@/_/[r]}
\end{array}
\quad = \quad + q^{-1}
\end{align*}
\quad = \quad [2] \quad := \quad q + q^{-1}
$$

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\[
\begin{align*}
\begin{array}{ccc}
\begin{tikzpicture}
\draw[->] (0,0) -- (1,1);
\draw[->] (0,1) -- (1,0);
\end{tikzpicture}
& = \begin{tikzpicture}
\draw (0,0) circle (1);
\end{tikzpicture} - q^{-1} & , & \begin{tikzpicture}
\draw (0,0) circle (1);
\draw[->] (0,0) -- (1,1);
\draw[->] (0,1) -- (1,0);
\end{tikzpicture}
& = -q \\
\end{array}
\end{align*}
\]

\[
\begin{tikzpicture}
\draw (0,0) circle (0.5);
\end{tikzpicture}
\begin{array}{c}
\text{= [2] := } q + q^{-1}
\end{array}
\]

i.e. as a function from the set of (oriented) knot diagrams to \( \mathbb{Z}[q, q^{-1}] \) that is invariant under the Reidemeister moves.
The Kauffman bracket

How to interpret $[D]$?

- As a rule for a “state sum” expansion for the Jones polynomial:
  
  $$
  [D] = [D] - q^{-1} [U] 
  = 0 - q^{-1} - q^{-1} U + q^{-2} 0 0 
  = [2](q + q^{-3})
  $$

- As a functor from the category of oriented tangles (knot pieces) to the category $\mathcal{T}L$ of $\mathbb{Z}[q, q^{-1}]$-linear combinations of planar curves (modulo the “circle relation”):

\[ \begin{array}{c}
\begin{array}{c}
\backslash
\end{array}
\end{array} \rightarrow 
\begin{array}{c}
\begin{array}{c}
\backslash
\end{array}
\end{array} - q \]
The Kauffman bracket

How to interpret \([\mathcal{D}]\)?

- As a rule for a “state sum” expansion for the Jones polynomial:

\[
\begin{align*}
\left[ \begin{array}{c}
\includegraphics{figure1.png}
\end{array} \right] &= \left[ \begin{array}{c}
\includegraphics{figure2.png}
\end{array} \right] - q^{-1} \left[ \begin{array}{c}
\includegraphics{figure3.png}
\end{array} \right] \\
&= \left[ \begin{array}{c}
\includegraphics{figure4.png}
\end{array} \right] - q^{-1} \left[ \begin{array}{c}
\includegraphics{figure5.png}
\end{array} \right] - q^{-1} \left[ \begin{array}{c}
\includegraphics{figure6.png}
\end{array} \right] + q^{-2} \left[ \begin{array}{c}
\includegraphics{figure7.png}
\end{array} \right] \\
&= [2](q + q^{-3})
\end{align*}
\]

- As a functor from the category of oriented tangles (knot pieces) to the category \(\mathcal{T}\mathcal{L}\) of \(\mathbb{Z}[q, q^{-1}]\)-linear combinations of planar curves (modulo the “circle relation”):
The Kauffman bracket

How to interpret $[\mathcal{D}]$?

- As a rule for a “state sum” expansion for the Jones polynomial:

$$
[\begin{array}{c}
\text{\textbullet} \\
\text{\textbullet}
\end{array}] = [\begin{array}{c}
\text{\textbullet} \\
\text{\textbullet}
\end{array}] - q^{-1} [\begin{array}{c}
\text{\textbullet}
\end{array}]
= [\begin{array}{c}
\text{\textbullet}
\end{array}] - q^{-1} [\begin{array}{c}
\text{\textbullet}
\end{array}] - q^{-1} [\begin{array}{c}
\text{\textbullet}
\end{array}] + q^{-2} [\begin{array}{c}
\text{\textbullet} \\
\text{\textbullet}
\end{array}]
= [2](q + q^{-3})
$$

- As a functor from the category of oriented tangles (knot pieces) to the category $\mathcal{T}\mathcal{L}$ of $\mathbb{Z}[q, q^{-1}]$-linear combinations of planar curves (modulo the “circle relation”):
The Kauffman bracket

How to interpret \([D]\)?

- As a rule for a “state sum” expansion for the Jones polynomial:
  \[
  \begin{align*}
  [\cell] &= [\cell] - q^{-1} [\cell] \\
  &= [\cell] - q^{-1} [\cell] - q^{-1} [\cell] + q^{-2} [\cell] \\
  &= [2](q + q^{-3})
  \end{align*}
  \]

- As a functor from the category of oriented tangles (knot pieces) to the category \(\mathcal{TL}\) of \(\mathbb{Z}[q, q^{-1}]\)-linear combinations of planar curves (modulo the “circle relation”):

\[\text{Diagram showing functor action.}\]
The Kauffman bracket

How to interpret $[D]$?

- As a rule for a “state sum” expansion for the Jones polynomial:

\[
\begin{align*}
[D] & = \boxed{\begin{array}{c}
\includegraphics{circle.png}
\end{array}} - q^{-1} \boxed{\begin{array}{c}
\includegraphics{arc.png}
\end{array}} \\
& = \boxed{\begin{array}{c}
\includegraphics{circle.png}
\end{array}} - q^{-1} \boxed{\begin{array}{c}
\includegraphics{arc.png}
\end{array}} - q^{-1} \boxed{\begin{array}{c}
\includegraphics{arc.png}
\end{array}} + q^{-2} \boxed{\begin{array}{c}
\includegraphics{circle.png}
\end{array}} \\
& = [2](q + q^{-3})
\end{align*}
\]

- As a functor from the category of oriented tangles (knot pieces) to the category $\mathcal{TL}$ of $\mathbb{Z}[q, q^{-1}]$-linear combinations of planar curves (modulo the “circle relation”):

\[
\begin{array}{c}
\includegraphics{tangle.png}
\end{array} \rightarrow \begin{array}{c}
\includegraphics{tangle.png}
\end{array} - q
\end{array}
\]
Diagrammatics for $\text{Rep}(U_q(\mathfrak{sl}_2))$

Q: What is the category $\mathcal{TL}$?
A: Back in 1932, Rummer, Teller, and Weyl knew the answer from their study of invariant vectors in tensor products of the standard representation $V = \mathbb{C}^2$ of $\mathfrak{sl}_2$: 
Diagrammatics for $\text{Rep}(U_q(\mathfrak{sl}_2))$

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Q: What is the category $\mathcal{T}L$?
A: Back in 1932, Rummer, Teller, and Weyl knew the answer from their study of invariant vectors in tensor products of the standard representation $\mathcal{V} = \mathbb{C}^2$ of $\mathfrak{sl}_2$:

\begin{equation}
1. \text{Fundamentalsatz annnehmen, daß die Invariante } J \text{ ein Monom ist, welches wir durch sein Valenzschema } S \text{ abbilden. Es bestehe aus } N \text{ Strichen zwischen den } n \text{ Punkten } x, y, \ldots, z. \text{ Wir stützen uns darauf, daß man mit Hilfe der Relation (2):}
\end{equation}

\begin{equation}
(3) \quad x \quad z \quad = \quad \begin{array}{c}
\quad \quad \\
\quad \quad \\
\quad \quad \\
\quad \quad \\
\quad \quad \\
\end{array}
\quad + \quad \begin{array}{c}
\quad \\
\quad \\
\quad \\
\quad \\
\quad \\
\end{array}
\end{equation}
Diagrammatics for $\text{Rep}(U_q(\mathfrak{sl}_2))$

Q: What is the category $\mathcal{T}\mathcal{L}$?

**Theorem (Folklore, Rummer-Teller-Weyl)**

The category $\mathcal{T}\mathcal{L}$ with objects $n \in \mathbb{N}$ and morphisms $n \to m$ consisting of $\mathbb{Z}[q, q^{-1}]$-linear combinations of $(m, n)$ planar curves, modulo the circle relation, is equivalent to the full subcategory of $\text{Rep}(U_q(\mathfrak{sl}_2))$ tensor generated by the standard representation.

- This is a diagrammatic incarnation of results of Reshetikhin-Turaev, that build a knot invariant $P_g(\mathcal{K}) \in \mathbb{Z}[q, q^{-1}]$ for each simple Lie algebra $g$. 
Diagrammatics for $\text{Rep}(U_q(\mathfrak{sl}_2))$

Q: What is the category $\mathcal{T}\mathcal{L}$?

Theorem (Folklore, Rummer-Teller-Weyl)

"$\mathcal{T}\mathcal{L}$ describes the category of $\mathfrak{sl}_2$ representations."

- This is a diagrammatic incarnation of results of Reshetikhin-Turaev, that build a knot invariant $P_g(K) \in \mathbb{Z}[q, q^{-1}]$ for each simple Lie algebra $\mathfrak{g}$. 
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Q: What is the category $\mathcal{T}\mathcal{L}$?

Theorem (Folklore, Rummer-Teller-Weyl)

“$\mathcal{T}\mathcal{L}$ describes the category of $\mathfrak{sl}_2$ representations.”

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Quantum $\mathfrak{sl}_n$ knot polynomials and webs

- Following work of Kuperberg, Murakami-Ohtsuki-Yamada (1998) give an analogue of the Kauffman bracket for the $\mathfrak{sl}_n$ knot polynomials:

$$
\left[ \begin{array}{c}
\downarrow & \uparrow \\
\end{array} \right]_n = \left[ \begin{array}{c}
\downarrow & \uparrow \\
\end{array} \right] - q^{-1}, \quad \left[ \begin{array}{c}
\downarrow & \uparrow \\
\end{array} \right]_n = -q
$$

- Cautis-Kamnitzer-Morrison define a category $n\text{Web}$ presented by generators and relations:

$$
\begin{array}{c}
k \\
\end{array}, \quad \begin{array}{c}
k+l \\
\end{array}, \quad \begin{array}{c}
k & l \\
\end{array}
$$
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\end{array}]_n &= \begin{array}{c}
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\end{array} - q^{-1} \\
\begin{array}{c}
\text{\textbullet} \\
\end{array} + \begin{array}{c}
\text{\textbullet} \\
\end{array}, \\
[\begin{array}{c}
\text{\textbullet} \\
\end{array}]_n &= -q \\
\begin{array}{c}
\text{\textbullet} \\
\end{array} + \begin{array}{c}
\text{\textbullet} \\
\end{array}.
\end{align*}
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\end{array}, \\
\begin{array}{c}
\text{\textbullet} \\
\end{array} + \begin{array}{c}
\text{\textbullet} \\
\end{array}, \\
\begin{array}{c}
\text{\textbullet} \\
\end{array} l,
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\[
\left[ \begin{array}{c}
\text{diagram 1} \\
\text{diagram 2}
\end{array} \right]_n = \text{diagram 3} - q^{-1} \text{diagram 4}, \quad \left[ \begin{array}{c}
\text{diagram 5} \\
\text{diagram 6}
\end{array} \right]_n = -q \text{diagram 7} + \text{diagram 8}
\]

- Cautis-Kamnitzer-Morrison define a category $n\text{Web}$ presented by generators and relations:

\[
\begin{array}{c}
\text{diagram 9} \\
\text{diagram 10} \\
\text{diagram 11}
\end{array}
\]

\[
\begin{array}{c}
\text{diagram 12} \\
\text{diagram 13} \\
\text{diagram 14}
\end{array}
\]

\[
\begin{array}{c}
\text{diagram 15} \\
\text{diagram 16} \\
\text{diagram 17}
\end{array}
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\text{Diagram 3} \\
\text{Diagram 4}
\end{array} \right] - q^{-1}, \\
\left[ \begin{array}{c}
\text{Diagram 5} \\
\text{Diagram 6}
\end{array} \right]_n &= -q
\end{align*}
$$

- Cautis-Kamnitzer-Morrison define a category $n\text{Web}$ presented by generators and relations:

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\begin{align*}
\left[ \begin{array}{c}
k+l \\
\text{Diagram 7}
\end{array} \right] &= \left[ \begin{array}{c}
k+l \\
\text{Diagram 8}
\end{array} \right]_{k+l}, \\
\left[ \begin{array}{c}
k+l+p \\
\text{Diagram 9}
\end{array} \right] &= \left[ \begin{array}{c}
k+l+p \\
\text{Diagram 10}
\end{array} \right]_{k+l+p}, \\
\left[ \begin{array}{c}
k+l+p \\
\text{Diagram 11}
\end{array} \right] &= \left[ \begin{array}{c}
k+l+p \\
\text{Diagram 12}
\end{array} \right]_{k+l+p}
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\text{ } \\
\end{array} \right]_n &= \left[ \begin{array}{c}
\text{ } \\
\text{ } \\
\end{array} \right] - q^{-1},
\left[ \begin{array}{c}
\text{ } \\
\text{ } \\
\end{array} \right]_n &= -q,
\left[ \begin{array}{c}
\text{ } \\
\text{ } \\
\end{array} \right] +
\end{align*}
$$

- Cautis-Kamnitzer-Morrison define a category $n\text{Web}$ presented by generators and relations:

$$
\begin{align*}
[\begin{array}{c}
\text{ } \\
\text{ } \\
\end{array}] & = \sum_{j=0}^{\min(a,b)} \binom{a-b+k-l}{j},
\end{align*}
$$
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- Following work of Kuperberg, Murakami-Ohtsuki-Yamada (1998) give an analogue of the Kauffman bracket for the $\mathfrak{sl}_n$ knot polynomials:

\[
\begin{aligned}
\left[ \begin{array}{c}
  \quad \quad \\
  \end{array} \right]_n &= \left[ \begin{array}{c}
  \quad \quad \\
  \end{array} \right] - q^{-1} \\
\left[ \begin{array}{c}
  \quad \quad \\
  \end{array} \right]_n &= -q \\
\end{aligned}
\]

- Cautis-Kamnitzer-Morrison define a category $n\text{Web}$ presented by generators and relations:

\[
\begin{align*}
\left[ \begin{array}{c}
  \quad \quad \\
  \end{array} \right] & = \left[ \begin{array}{c}
  \quad \quad \\
  \end{array} \right] - q^{-1} \\
\left[ \begin{array}{c}
  \quad \quad \\
  \end{array} \right] & = -q \\
\end{align*}
\]
Quantum $\mathfrak{sl}_n$ knot polynomials and webs

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$$
\begin{align*}
\left[ \begin{array}{c}
\uparrow \\
\downarrow \\
\end{array} \right]_n &= \left[ \begin{array}{c}
\uparrow \\
\downarrow \\
\end{array} \right] - q^{-1} \\
\left[ \begin{array}{c}
\uparrow \\
\downarrow \\
\end{array} \right]_n &= -q \\
\left[ \begin{array}{c}
\uparrow \\
\downarrow \\
\end{array} \right]_n &= +
\end{align*}
$$

**Theorem** (Cautis-Kamnitzer-Morrison, 2012)

$n\text{Web}$ is equivalent to the subcategory of $\text{Rep}(U_q(\mathfrak{gl}_n))$ tensor generated by the fundamental representations $\wedge^k V$.

- (Conceptual aside: we expect that such a 2d “generators-and-relations” presentation should exist since $\text{Rep}(U_q(\mathfrak{gl}_n))$ is a monoidal category.)
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- Following work of Kuperberg, Murakami-Ohtsuki-Yamada (1998) give an analogue of the Kauffman bracket for the $\mathfrak{sl}_n$ knot polynomials:

$$
\left[ \begin{array}{c}
\uparrow \\
\downarrow \\
\end{array} \right]_n = \left[ \begin{array}{c}
\uparrow
\downarrow
\end{array} \right] - q^{-1}, \quad 
\left[ \begin{array}{c}
\downarrow
\uparrow
\end{array} \right]_n = -q \left[ \begin{array}{c}
\downarrow
\uparrow
\end{array} \right] + \left[ \begin{array}{c}
\downarrow
\uparrow
\end{array} \right]
$$

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Khovanov homology

- Khovanov uses the diagrammatic description of $V_q(\mathcal{K})$ to define a homology theory $\text{Kh}(\mathcal{K})$ for knots that categorifies the Jones polynomial, i.e. $\dim(\text{Kh}(\mathcal{K})) = V_q(\mathcal{K})$.
- $\text{Kh}(\mathcal{K})$ has had spectacular applications in 3- and 4-dimensional topology (unknot detection, slice genus bounds, concordance invariants)
- Khovanov homology is functorial with respect to cobordisms $\Sigma \subset S^3 \times [0, 1]$:
  
  ![Diagram](image)

  i.e. it is inherently 4-dimensional.
- We thus expect that the theory should be described by 3-dimensional diagrammatics...
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- Khovanov homology is functorial with respect to cobordisms $\Sigma \subset S^3 \times [0, 1]$:

$$\begin{align*}
\text{Kh} \left( \begin{array}{c}
\infty
\end{array} \right) \xrightarrow{\text{Kh}(\Sigma)} \text{Kh} \left( \begin{array}{c}
\infty
\end{array} \right)
\end{align*}$$

i.e. it is inherently 4-dimensional.

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- Khovanov homology is functorial with respect to cobordisms $\Sigma \subset S^3 \times [0, 1]$:

\[
\begin{array}{c}
\begin{array}{c}
  \includegraphics[width=0.3\textwidth]{khovanov_diagram.png}
  \\
  \leftrightarrow
  \\
  \text{Kh}\left(\includegraphics[width=0.1\textwidth]{khovanov_diagram.png}\right) \xrightarrow{\text{Kh}(\Sigma)} \text{Kh}\left(\includegraphics[width=0.1\textwidth]{khovanov_diagram.png}\right)
\end{array}
\end{array}
\]

i.e. it is inherently 4-dimensional.
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$$\begin{array}{ccc}
\text{Kh}(\bigcirc) & \xrightarrow{\text{Kh}(\Sigma)} & \text{Kh}(\bigcirc) \\
\text{i.e. it is inherently 4-dimensional.}
\end{array}$$

- We thus expect that the theory should be described by 3-dimensional diagrammatics...
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  \[
  \begin{pmatrix}
  \begin{array}{c}
  \includegraphics[width=0.4\textwidth]{khovanov_diagram}
  \\
  \end{array}
  \end{pmatrix}
  \longrightarrow
  \begin{pmatrix}
  \begin{array}{c}
  \includegraphics[width=0.4\textwidth]{khovanov_diagram}
  \\
  \end{array}
  \end{pmatrix}
  \xrightarrow{Kh(\Sigma)}
  \begin{pmatrix}
  \begin{array}{c}
  \includegraphics[width=0.4\textwidth]{khovanov_diagram}
  \\
  \end{array}
  \end{pmatrix}
  \]

  i.e. it is inherently 4-dimensional.
- We thus expect that the theory should be described by 3-dimensional diagrammatics...
3d diagrammatics for Khovanov homology

- Khovanov’s invariant is given by promoting the Kauffman bracket one dimension higher, using homological algebra:

\[
\left[ \begin{array}{c}
\includegraphics[width=0.2\textwidth]{knot_diagram}
\end{array} \right] = \frac{\partial}{\partial q} - q^{-1}
\]

- Following Bar-Natan, we interpret this as a chain complex in a certain (additive, monoidal) 2-category $\mathcal{BN}$ of curves and decorated surfaces, modulo local relations given diagrammatically as:

\[
\begin{align*}
\includegraphics[width=0.4\textwidth]{relations_diagram}
\end{align*}
\]

- These relations encode the Frobenius algebra structure on $H^*(\mathbb{C}P^1)$, and $\mathcal{BN}$ “categorifies” $\mathcal{T}\mathcal{L}$. 
3d diagrammatics for Khovanov homology

- Khovanov’s invariant is given by promoting the Kauffman bracket one dimension higher, using homological algebra:

\[
\begin{align*}
\left[\begin{array}{c}
\uparrow & \\
\downarrow & \\
\end{array}\right] &= \left[\begin{array}{c}
\circlearrowleft \\
\circlearrowright
\end{array}\right] - q^{-1}
\end{align*}
\]

- Following Bar-Natan, we interpret this as a chain complex in a certain (additive, monoidal) 2-category \(\mathcal{BN}\) of curves and decorated surfaces, modulo local relations given diagrammatically as

\[
\begin{align*}
\text{cylinder} &= \text{cylinder} + \text{Disk} \quad \Rightarrow 0 \\
\text{Disk} &= 1 \\
\text{Cube} &= 0
\end{align*}
\]

- These relations encode the Frobenius algebra structure on \(H^*(\mathbb{CP}^1)\), and \(\mathcal{BN}\) “categorifies” \(\mathcal{T}\mathcal{L}\).
3d diagrammatics for Khovanov homology

- Khovanov’s invariant is given by promoting the Kauffman bracket one dimension higher, using homological algebra:

\[
\begin{array}{c}
\begin{tikzpicture}
\draw [->] (0,0) -- (2,0);
\draw [->] (2,0) -- (4,0);
\draw (0,0) -- (0,2);
\draw (2,0) -- (2,2);
\end{tikzpicture}
\end{array}
\begin{tikzpicture}
\draw (0,0) -- (1,1);
\draw (1,0) -- (0,1);
\draw (2,0) -- (1,1);
\draw (1,0) -- (2,1);
\draw (2,0) -- (2,1);
\end{tikzpicture}
\xrightarrow{q^{-1}}
\end{array}
\]

- Following Bar-Natan, we interpret this as a chain complex in a certain (additive, monoidal) 2-category \(\mathcal{BN}\) of curves and decorated surfaces, modulo local relations given diagrammatically as

\[
\begin{align*}
\begin{tikzpicture}
\draw (0,0) -- (1,1);
\draw (1,0) -- (0,1);
\end{tikzpicture} & \quad + \\
\begin{tikzpicture}
\draw (0,0) -- (1,1);
\draw (1,0) -- (0,1);
\end{tikzpicture} & \quad = \\
\begin{tikzpicture}
\draw (0,0) -- (1,1);
\draw (1,0) -- (0,1);
\end{tikzpicture} & \quad = \\
\begin{tikzpicture}
\draw (0,0) -- (1,1);
\draw (1,0) -- (0,1);
\end{tikzpicture} & \quad = \\
\begin{tikzpicture}
\draw (0,0) -- (1,1);
\draw (1,0) -- (0,1);
\end{tikzpicture} & \quad = 0
\end{align*}
\]

- These relations encode the Frobenius algebra structure on \(H^*(\mathbb{C}P^1)\), and \(\mathcal{BN}\) “categorifies” \(\mathcal{TL}\).
3d diagrammatics for Khovanov homology

- Khovanov’s invariant is given by promoting the Kauffman bracket one dimension higher, using homological algebra:

\[
\begin{align*}
\left[ & \begin{array}{c}
\uparrow \\
\downarrow \\
\end{array} \right] = & \quad \begin{array}{c}
\bigcup \\
\end{array} \quad q^{-1} \\
\end{align*}
\]

- Following Bar-Natan, we interpret this as a chain complex in a certain (additive, monoidal) 2-category \(\mathcal{BN}\) of curves and decorated surfaces, modulo local relations given diagrammatically as:

\[
\begin{align*}
\text{cylinder} & = \text{two cylinders} + \text{two caps}, \\
\text{cylinder} & = 0, \\
\text{caps} & = 1, \\
\text{rectangle} & = 0
\end{align*}
\]

- These relations encode the Frobenius algebra structure on \(H^*(\mathbb{C}P^1)\), and \(\mathcal{BN}\) “categorifies” \(\mathcal{T}\).
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- Khovanov’s invariant is given by promoting the Kauffman bracket one dimension higher, using homological algebra:

\[
\left[ \begin{array}{c}
\uparrow \\
\downarrow
\end{array} \right] = \bigcup \xrightarrow{q^{-1}}
\]

- Following Bar-Natan, we interpret this as a chain complex in a certain (additive, monoidal) 2-category \( \mathcal{BN} \) of curves and decorated surfaces, modulo local relations given diagrammatically as

\[
\begin{align*}
\text{cylinder} & = \text{disk} + \text{disk} , \\
\text{generator} & = 0 , \\
\text{generator} & = 1 , \\
\text{identity} & = 0
\end{align*}
\]

- These relations encode the Frobenius algebra structure on \( H^*(\mathbb{CP}^1) \), and \( \mathcal{BN} \) “categorifies” \( \mathcal{T}\mathcal{L} \).
3d diagrammatics for Khovanov homology

- Khovanov’s invariant is given by promoting the Kauffman bracket one dimension higher, using homological algebra:

\[
\begin{align*}
[\includegraphics[width=1cm]{diagram1.png}] &= \includegraphics[width=2cm]{diagram2.png} \\
&\quad \quad \quad q^{-1}
\end{align*}
\]

- Following Bar-Natan, we interpret this as a chain complex in a certain (additive, monoidal) 2-category \(\mathcal{BN}\) of curves and decorated surfaces, modulo local relations given diagrammatically as

\[
\begin{align*}
\includegraphics[width=3cm]{diagram3.png} &= \includegraphics[width=3cm]{diagram4.png} + \includegraphics[width=3cm]{diagram5.png} \\
\includegraphics[width=3cm]{diagram6.png} &= 0 \\
\includegraphics[width=3cm]{diagram7.png} &= 1 \\
\includegraphics[width=3cm]{diagram8.png} &= 0
\end{align*}
\]

- These relations encode the Frobenius algebra structure on \(H^*(\mathbb{CP}^1)\), and \(\mathcal{BN}\) “categorifies” \(\mathcal{T\mathcal{L}}\).
Khovanov-Rozansky homology and foams

- Khovanov-Rozansky define a knot homology $\text{KhR}_n(\mathcal{K})$ that categorifies the $\mathfrak{sl}_n$ knot polynomials. These invariants enjoy applications and properties similar to those of $\text{Kh}(\mathcal{K})$ (and refined versions thereof).

- In joint work with H. Queffelec (building on earlier work of Khovanov, Rozansky, and Mackaay-Stotic-Vaz) we construct a 3d diagrammatic 2-category $n\text{Foam}$ that categorifies $n\text{Web}$ and allows for a 3d diagrammatic construction of $\text{KhR}_n(\mathcal{K})$. 
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\[
= \sum_{\alpha} (-1)^{|\hat{\alpha}|}
\]

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\[
\begin{align*}
\begin{array}{c}
\begin{tikzpicture}
  \draw[->, thick] (0,0) -- (0.5,0);
  \draw[->, thick] (0,0.5) -- (0.5,0.5);
  \draw[->, thick] (0,1) -- (0.5,1);
  \draw[->, thick] (0,1.5) -- (0.5,1.5);
\end{tikzpicture}
\end{array}
\end{align*}
\quad =
\begin{array}{c}
\begin{tikzpicture}
  \draw[->, thick] (0,0) -- (0.5,0);
  \draw[->, thick] (0,0.5) -- (0.5,0.5);
  \draw[->, thick] (0,1) -- (0.5,1);
  \draw[->, thick] (0,1.5) -- (0.5,1.5);
  \fill[red!50] (0,0.75) rectangle (0.5,1.25);
\end{tikzpicture}
\end{array}
\quad \rightarrow q^{-1}
\]

\[
\begin{align*}
\begin{array}{c}
\begin{tikzpicture}
  \draw[->, thick] (0,0) -- (0.5,0);
  \draw[->, thick] (0,0.5) -- (0.5,0.5);
  \draw[->, thick] (0,1) -- (0.5,1);
  \draw[->, thick] (0,1.5) -- (0.5,1.5);
\end{tikzpicture}
\end{array}
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\quad =
\begin{array}{c}
\begin{tikzpicture}
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  \draw[->, thick] (0,0.5) -- (0.5,0.5);
  \draw[->, thick] (0,1) -- (0.5,1);
  \draw[->, thick] (0,1.5) -- (0.5,1.5);
  \fill[red!50] (0,0.75) rectangle (0.5,1.25);
\end{tikzpicture}
\end{array}
\quad \rightarrow
\begin{array}{c}
\begin{tikzpicture}
  \draw[->, thick] (0,0) -- (0.5,0);
  \draw[->, thick] (0,0.5) -- (0.5,0.5);
  \draw[->, thick] (0,1) -- (0.5,1);
  \draw[->, thick] (0,1.5) -- (0.5,1.5);
\end{tikzpicture}
\end{array}
\quad q
\]

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An application: structures on annular knot invariants

- It is of topological interest to extend various knot homologies to knots in 3-manifolds $\mathcal{M} \neq S^3$.
- In work with Queffelec (following work of Asaeda-Przytycki-Sikora), we use foams embedded in a thickening of the annulus $A$ to construct analogues $\mathcal{A}KhR_n(\mathcal{K})$ of Khovanov-Rozansky homology for knots $\mathcal{K} \subset A \times [0, 1]$.
- We further show that such knot homologies can be described in terms of annular foams that are rotationally symmetric.
- Taking a radial slice, such foams correspond to $\mathfrak{sl}_n$ webs, i.e. to maps between $\mathfrak{sl}_n$ representations.
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• In work with Queffelec (following work of Asaeda-Przytycki-Sikora), we use foams embedded in a thickening of the annulus $\mathcal{A}$ to construct analogues $\mathcal{AKhR}_n(\mathcal{K})$ of Khovanov-Rozansky homology for knots $\mathcal{K} \subset \mathcal{A} \times [0, 1]$.

Theorem (Grigsby-Licata-Wehrli, Queffelec-R.)

The annular knot invariant $\mathcal{AKhR}_n(\mathcal{K})$ carries an action of $\mathfrak{sl}_n$.

⇒ invariants of knotted surfaces in 4d.

• We further show that such knot homologies can be described in terms of annular foams that are rotationally symmetric.

• Taking a radial slice, such foams correspond to $\mathfrak{sl}_n$ webs, i.e. to maps between $\mathfrak{sl}_n$ representations.
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**Theorem (Beliakova-Putyra-Wehrli, Putyra-R.)**

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Foams and representation theory

The 2-category $n\text{Foam}$ lives at the intersection of low-dimensional topology and categorical representation theory.

- On the one hand, $n\text{Foam}$ (and its annular variant) are the setting for various knot homology theories.
- On the other, taking the $n \to \infty$ limit gives a 2-category that is a 3d diagrammatic model for the 2-category of singular Soergel bimodules:

\[
\begin{array}{c}
\vdots \\
\begin{array}{c}
\text{...}
\end{array} \\
\begin{array}{c}
\text{...}
\end{array} \\
\end{array}
\leftrightarrow \mathbb{C}[x_1, \ldots, x_m] \otimes \mathbb{C}[x_1, \ldots, x_i+x_{i+1}, x_i x_{i+1}, \ldots, x_m] \mathbb{C}[x_1, \ldots, x_m]
\]

The latter is a certain 2-category of bimodules over polynomial rings $\mathbb{C}[x_1, \ldots, x_m]^{W_j}$ equivalent to a 2-category of perverse sheaves on products of partial flag varieties (the “Hecke category” is a full 2-subcategory).

- Results of Rouquier assign a complex of (singular) Soergel bimodules to any braid, and Khovanov builds triply-graded (Khovanov-Rozansky) knot homology using the Hochschild cohomology of these bimodules.
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\cdots & \leq & \cdots \\
\& & \\
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$$f \mapsto x_1 f \otimes 1 - f \otimes x_2$$

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$$\begin{align*}
\begin{matrix}
& \quad & \quad & \\
& \quad & \quad & \\
& \quad & \quad & \\
\end{matrix}
\quad =
\quad \begin{tikzpicture}
\begin{scope}[xscale=-1]
\draw[->] (0,0) .. controls (0.5,0.25) and (1,0) .. (1.5,0);
\draw[->] (0,0) .. controls (0.5,-0.25) and (1,0) .. (1.5,0);
\end{scope}
\draw[->] (3,0) .. controls (3.5,0.1) and (4,0.5) .. (4,1);
\draw[->] (3,0) .. controls (3.5,-0.1) and (4,-0.5) .. (4,1);
\filldraw[blue!50!white] (0,0) circle (0.25);
\filldraw[blue!50!white] (1,0) circle (0.25);
\filldraw[blue!50!white] (2,0) circle (0.25);
\filldraw[blue!50!white] (3,0) circle (0.25);
\filldraw[blue!50!white] (4,0) circle (0.25);
\end{tikzpicture}
\quad q^{-1}
\end{align*}$$

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An application: knot homology in a handlebody

- We obtain a complex of singular Soergel bimodules associated to any braided web, and can apply Hochschild cohomology:

- The complement of the web is a handlebody $\mathcal{H}_g$ (i.e. “the inside” of a genus $g$ surface), so this diagram describes a knot $\mathcal{K} \subset \mathcal{H}_g$. 
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Theorem (R.-Tubbenhauer)

There exists a homology theory for knots in genus $g$ handlebodies, that extends triply-graded (Khovanov-Rozansky) knot homology.
Thanks!